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Encounter of an asteroid with a planet

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This paper describes the trajectory of an asteroid (or a comet or spacecraft) as it approaches a planet of much greater mass. The solution of this two-body problem is an instructive first approximation to more refined treatments that include the gravitational forces of the Sun and of planets other than the target planet. Detailed properties of encounter trajectories are derived. As an illustration, it is shown that the collision cross section of the planet is greater by a factor $F = 1 + (v_{\text{escape}}/v_0)^2$ compared to its cross section in the absence of gravitational forces, where $v_{\text{escape}}$ is the minimal escape speed from the surface of the planet and $v_0$ is the approach speed of the asteroid at an effectively infinite distance. Sample values of $F$ are given for Earth, Mars, Jupiter, and Saturn. © 2006 American Association of Physics Teachers. [DOI: 10.1119/1.2203644]

I. INTRODUCTION

One purpose of this paper is to identify the conditions under which an approaching asteroid will collide with the Earth or another planet. The analysis can also be applied to other problems in celestial mechanics, such as the process by which the heliocentric orbit of an asteroid or comet is altered, perhaps profoundly (for example, from an hyperbola to an ellipse or vice versa), during an encounter with a planet. Other applications include gravitational assist by which the encounter of a spacecraft with a planet can be used to increase the heliocentric speed of the spacecraft and make possible a high-speed trajectory to a much more distant planet (for example, a flyby of Jupiter enroute to Pluto). A similar technique can be used to modify the planetary orbit of a spacecraft by its flyby of a satellite of the planet (for example, the Cassini spacecraft in orbit about Saturn).

As an asteroid of mass $m$ approaches a planet of mass $M$, where $m \ll M$, the gravitational attraction of the planet accelerates the asteroid along the line from the moving asteroid to the center of the planet. The speed of the asteroid is increased during the approach and its trajectory is bent inward toward the planet. In this paper, the gravitational attraction of the Sun during the encounter is neglected and the encounter is treated as a two-body problem (asteroid and planet), with the motion of the asteroid described in a coordinate system centered on the planet. Because the gravitational force on the asteroid is proportional to its mass, the magnitude and direction of its acceleration are independent of its mass and the resulting trajectory is the same for any mass $m$, provided that $m \ll M$. The initial conditions are that the asteroid is at an effectively infinite radial distance $r_0$ from the planet and is moving at velocity $v_0$ along a line that misses the center of the planet by a perpendicular distance $b$ as shown in Fig. 1.

II. BASIC PHYSICS OF THE ENCOUNTER

There is a plane that passes through the initial position of the asteroid and the center of the planet and also contains the vector $v_0$. The radially inward gravitational force on the asteroid and the consequential acceleration vector have no component perpendicular to this plane. This situation continues to be the case as the asteroid moves inward. Hence, its entire subsequent trajectory lies in this plane, termed the encounter plane.

By the same reasoning, the vector angular momentum per unit mass $\ell$ of the asteroid, with respect to the center of the planet (assumed spherically symmetrical), is perpendicular to the encounter plane and is a constant of the motion with magnitude

$$\ell = v_0 b.$$ (1)

The total energy per unit mass of the asteroid $E$ is also a constant of the motion and is given by

$$E = T + U,$$ (2)

where $T$ is its kinetic energy per unit mass and $U$ is its potential energy per unit mass in the gravitational field of the planet. By Newton’s law of gravitation

$$U = -\frac{GM}{r},$$ (3)

where $r$ is the radial distance from the center of the planet to the asteroid and $G$ is Newton’s universal gravitational constant. Hence,

$$v^2 = v_0^2 + \frac{2GM}{r},$$ (4)

where $v$ is the speed of the asteroid at a radial distance $r$.

III. ANALYTICAL SOLUTION

It is shown in texts on mechanics\(^1\)\(^2\) that the resulting trajectory of the asteroid is an hyperbola whose focus is at the center of the planet. The equation of the hyperbola in polar coordinates $(r, \theta)$ with the origin at the center of the planet is

$$r = \frac{\ell^2/GM}{(1 + e \cos \theta)},$$ (5)

with the eccentricity

$$e = \left[1 + \frac{2E \ell^2}{(GM)^2}\right]^{1/2}.$$ (6)

The closest approach of the asteroid to the center of the planet, the periapsidal radial distance $r_p$, occurs when $\theta = 0^\circ$ in Eq. (5). Hence
The resulting quadratic equation has the solution for the eccentricity of the hyperbola as defined by the velocity $v_0$ at infinity and the parameter $b$. In this example $v_0 = 14.59 \text{ km/s}$, $e = 1.20$, $b/R = 4.20$, $\alpha = 153.23^\circ$, and the asymptote intersects the axis of symmetry at $r/R = 9.33$. All dimensions in this diagram are in units of the planet’s radius.

$$\ell^2 = \frac{r_p(1 + e)}{GM},$$

and the equation of the trajectory can be rewritten as

$$r = \frac{r_p(1 + e)}{1 + e \cos \theta},$$

with $r > 0$. On purely geometrical grounds, negative values of $r$ trace out the left-to-right mirrored branch of the two-branch hyperbola. The distance between the foci of the two branches as derived from Eq. (8) is

$$r_p = \frac{(e + 1)}{e - 1} + r_p = \frac{2r_p e}{e - 1}.$$  

By symmetry, the asymptotes of the two branches intersect the horizontal axis of Fig. 1 at one-half of the distance given by Eq. (9), namely at

$$r_p = \frac{e}{e - 1}.$$  

from the center of the planet.

Equation (6) can be rewritten in terms of the initial conditions by replacing $2E$ by $v_0^2$ and $\ell$ by $v_0 \beta$ to yield

$$e = \left[1 + \frac{v_0^2 \beta^2}{(GM)^2}\right]^{1/2}.$$  

A useful alternative expression for the eccentricity can be found from Eq. (6) by using Eq. (7) and replacing $2E$ by $v_0^2$. The resulting quadratic equation has the solution for $e > 1$

$$e = 1 + \frac{r_p v_0^2}{GM}.$$  

At the initial condition, $r$ in Eq. (8) is infinite and the approach asymptote intersects the axis of symmetry of the hyperbola at the angle

$$\alpha = \arccos\left(-\frac{1}{e}\right).$$  

The combination of Eqs. (10) and (13) yields the useful relation

$$b = r_p \left(\frac{e + 1}{e - 1}\right)^{1/2}.$$  

Equations (10), (13), and (14) are general and can be checked numerically with the help of Fig. 1, which is drawn for a specific case.

For a trajectory that misses the planet, it is readily found that during the entire encounter the velocity vector of the asteroid is rotated by the angle $(2\alpha - 180^\circ)$, a result that was used in the author’s previous paper.

**IV. THE COLLISION CASE**

Depending on the magnitudes of $M$, $v_0$, $b$, and the radius of the planet $R$, the trajectory of the asteroid either misses the planet ($r_p R$) or collides with it ($r_p/R$). When $r_p = R$, the asteroid makes a tangential collision with the planet. The corresponding value of $\pi b^2$ is the collision cross section of the planet. Thus, the collision cross section is larger than it would be in the absence of gravitational attraction by the factor

$$F = \frac{\pi b^2}{\pi R^2} = \left(\frac{b}{R}\right)^2,$$

where $b$ is the value of the parameter for a tangential collision.

At periapsis (closest approach to the planet) of the hyperbolic trajectory, the velocity vector $v_x$ of the asteroid is orthogonal to the radial vector from the center of the planet. The following equation combines conservation of energy and conservation of angular momentum at periapsis:

$$v_x = \frac{v_0}{\sqrt{1 + e}}.$$  

Fig. 2. A family of four encounter trajectories to the planet Jupiter for $v_0 = 14.59 \text{ km/s}$. The four approach asymptotes that are parallel and in the same plane, with $b/R = 8.40, 6.30, 4.20,$ and $2.10$, respectively, from right to left. The left-most trajectory results in a collision with the planet, but is drawn as though the entire mass of the planet were concentrated at its center. All dimensions in the diagram are in units of the planet’s radius as in Fig. 1. Note that the axes of symmetry of the several trajectories rotate counterclockwise as $b$ increases.
for several planets and calculated values of $F$ for $v_0 = 10$ km/s, a representative value. Values of $F$ for other values of $v_0$ can be calculated using Eq. (17). The values of $GM$ in km$^3$/s$^2$, the equatorial radius $R$ in km, and $v_{\text{escape}}$ from the equator in km/s are from Cox. The values of $R$ and $v_{\text{escape}}$ would be slightly different if attributed to spherical planets of the same mass and volume.

V. COMMENT

The two-body analysis of the present paper is instructive and often serves as a good first approximation, but the reader is cautioned to be aware of the refinements required by a full treatment of the multibody problem involving the Sun and other planets.

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The author is indebted to Christine Stevens for preparation of numerous revisions of this paper and to Joyce Chrsinger for publishable versions of the figures.

Table I. Data for several planets and calculated values of $F$.

<table>
<thead>
<tr>
<th>Planet</th>
<th>$GM$ (km$^3$/s$^2$)</th>
<th>$R$ (km)</th>
<th>$v_{\text{escape}}$ (km/s)</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth</td>
<td>$3.986 \times 10^5$</td>
<td>6378</td>
<td>11.18</td>
<td>2.25</td>
</tr>
<tr>
<td>Mars</td>
<td>$4.283 \times 10^4$</td>
<td>3397</td>
<td>5.02</td>
<td>1.25</td>
</tr>
<tr>
<td>Jupiter</td>
<td>$1.267 \times 10^8$</td>
<td>71 492</td>
<td>59.54</td>
<td>36.45</td>
</tr>
<tr>
<td>Saturn</td>
<td>$3.793 \times 10^7$</td>
<td>60 268</td>
<td>35.49</td>
<td>13.60</td>
</tr>
</tbody>
</table>

\[
v_{\text{escape}}^2 = v_0^2 + \frac{2GM}{R} = \left( \frac{bv_0}{R} \right)^2,
\]

so that

\[
F = 1 + \left( \frac{v_{\text{escape}}}{v_0} \right)^2,
\]

where

\[
v_{\text{escape}} = \sqrt{\frac{2GM}{R}}
\]

is the minimal escape speed from the surface of the planet.

Equation (17) is well known to workers in celestial mechanics and is worthy of broader attention from teachers and students of physics who are interested in past and future collisions of asteroids with the Earth and other planets.

Figure 1 shows a sample case of a tangential collision, that is, $r_f = R$, and Fig. 2 shows a family of four trajectories headed toward the same planet. Table I gives some basic data